

Last Time: Introduction to Linear maps.
w/ many examples

Recall: Let B be a basis of vector space V .
Let W be a vector space. Every function
 $f: B \rightarrow W$ extends (linearly) to a linear map
 $F: V \rightarrow W$ via the formula

$$F\left(\sum_{i=1}^n c_i b_i\right) = \sum_{i=1}^n c_i f(b_i).$$

Point: Linear maps are determined by where they send
a basis of the domain space.

More on Linear Maps

Let $L: V \rightarrow W$ be a linear map. The
kernel of L is $\ker(L) := \{v \in V : \underline{L(v) = 0_W}\}$
The range of L is $\text{ran}(L) := \{L(v) : v \in V\}$.

NB: $\ker(L) \subseteq V$ while $\text{ran}(L) \subseteq W$.

Prop: The kernel of L is subspace of $\text{dom}(L)$.

Pf: Let $L: V \rightarrow W$ be a linear map. We'll use
the subspace test to verify $\ker(L) \leq V$. Note

$$L(0_V) = L(0 \cdot 0_V) = 0 \cdot L(0_V) = 0_W,$$

so $[0_V \in \ker(L) \neq \emptyset]$ Now suppose $u, v \in \ker(L)$
and $c \in \mathbb{R}$. Now we apply L to $u + cv$:

$$L(u+cv) = L(u) + L(cv) = L(u) + cL(v) = 0_u + c \cdot 0_v = 0_w$$

Hence $[u+cv \in \ker(L)]$. Hence, by the subspace

Test we have $\ker(L) \leq V$. \square

Ex: Compute $\ker(L)$ for $L: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ w/ $L\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x \\ y+z \end{pmatrix}$.

Sol: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker(L)$ iff $L\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iff $\begin{pmatrix} x \\ y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

iff $\begin{cases} x = 0 \\ y+z = 0 \end{cases}$

Solving the corresponding linear system: $x=0, y=-z$

$$\therefore \ker(L) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : \underline{x=0}, \underline{y=-z} \right\}$$

$$= \left\{ \begin{pmatrix} 0 \\ -z \\ z \end{pmatrix} \in \mathbb{R}^3 : z \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}. \quad \square$$

NB: We computed a basis for $\ker(L)$, namely $\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$...

Ex: Compute $\ker(L)$ where $L: \mathcal{P}_2(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$

is given by $L(c + bx + ax^2) = \begin{pmatrix} 3a-b & 2b+c \\ a-c & a+b+c \end{pmatrix}$

Sol: $c + bx + ax^2 \in \ker(L)$

iff $L(c + bx + ax^2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

iff $\begin{pmatrix} 3a-b & 2b+c \\ a-c & a+b+c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

iff $\begin{cases} 3a-b & = 0 \\ 2b+c & = 0 \\ a-c & = 0 \\ a+b+c & = 0 \end{cases}$

Solving this linear system:

$$\begin{bmatrix} 3 & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 3 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \\ 0 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \\ d = 0 \end{cases}$$

$$\therefore \text{Ker}(L) = \{0 + 0x + 0x^2\} = \{0\}. \quad \square$$

Ex: Compute $\text{Ker}(L)$ for $L: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ given by
 $L \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix}.$

Sol: $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \text{Ker}(L)$ iff $L \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
 iff $\begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
 iff $\begin{cases} x+y+z = 0 \\ x-y+w = 0 \end{cases} \leftarrow \text{Solve!}$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \rightarrow \begin{cases} x + \frac{1}{2}z + \frac{1}{2}w = 0 \\ y + \frac{1}{2}z - \frac{1}{2}w = 0 \end{cases}$$

$$\therefore \begin{cases} x = -\frac{1}{2}s - \frac{1}{2}t \\ y = -\frac{1}{2}s + \frac{1}{2}t \\ z = s \\ w = t \end{cases} \quad \therefore \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}s - \frac{1}{2}t \\ -\frac{1}{2}s + \frac{1}{2}t \\ s \\ 0s + 1t \end{pmatrix} = s \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \ker(L) = \text{span} \left\{ \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ 1/2 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \square$$

What about the range space?

Prop: Suppose $L: V \rightarrow W$ is a linear map. Then $\text{ran}(L) \leq W$.

pf: Let $L: V \rightarrow W$ be a linear map. We apply the Subspace Test. Note $L(0_V) = 0_W \in \text{ran}(L) \neq \emptyset$.

Given $u, v \in \text{ran}(L)$ and $c \in \mathbb{R}$, write

$$u = \underline{L(\alpha)} \text{ and } v = \underline{L(\beta)} \text{ for some } \underline{\alpha, \beta \in V}.$$

$$\text{Note } u + cv = L(\alpha) + cL(\beta) = L(\alpha) + L(c\beta) = L(\alpha + c\beta).$$

But $\alpha + c\beta \in V$, so $u + cv = L(\alpha + c\beta)$ yields

$u + cv \in \text{ran}(L)$. Hence $\text{ran}(L) \leq W$ by the Subspace test. \square

NB: You can adapt this proof to show that, given

$$L: V \rightarrow W \text{ and } U \leq V, \quad L(U) := \{L(u) : u \in U\} \leq W \dots$$

Ex: Compute $\text{ran}(L)$ where $L: \mathcal{P}_2(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$

$$\text{is given by } L(c + bx + ax^2) = \begin{pmatrix} 3a - b & 2b + c \\ a - c & a + b + c \end{pmatrix}$$

$$\begin{aligned} \text{Sol: } \text{ran}(L) &= \{L(v) : v \in V\} \\ &= \left\{ \begin{pmatrix} 3a - b & 2b + c \\ a - c & a + b + c \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \end{aligned}$$

$$= \left\{ \begin{pmatrix} 3a & 0 \\ a & a \end{pmatrix} + \begin{pmatrix} -b & 2b \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & c \\ -c & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

$$= \left\{ a \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix} + b \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} + c \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

$$\therefore \text{ran}(L) = \text{span} \left\{ \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \right\}$$

NB: Earlier we showed this set is Lin indep. also \checkmark

Ex: Compute $\text{ran}(L)$ for $L: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ w/

$$L \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix}.$$

$$\underline{\text{Sol}}: \text{ran}(L) = \{ L(v) : v \in \mathbb{R}^4 \}$$

$$= \left\{ \begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix} : x, y, z, w \in \mathbb{R} \right\}$$

$$= \left\{ x \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -1 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} 0 \\ 1 \end{pmatrix} : x, y, z, w \in \mathbb{R} \right\}$$

$$\therefore \text{ran}(L) = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

not a basis!

(e.g. $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$)

Exercise: Compute a basis from $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ for $\text{ran}(L)$. \checkmark

Up until now: line $\ker(L) \leq V$ and $\text{ran}(L)$

"kernel of L " / "null space of L "

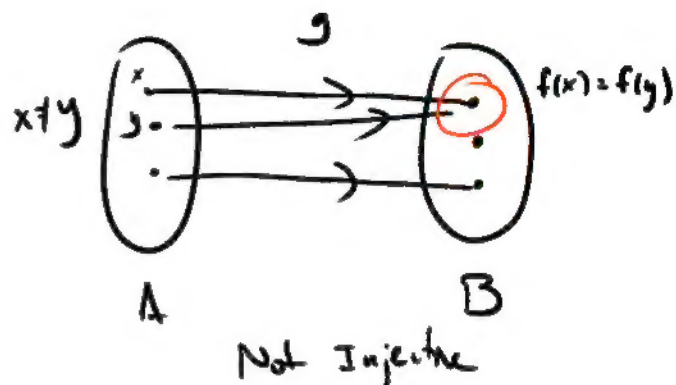
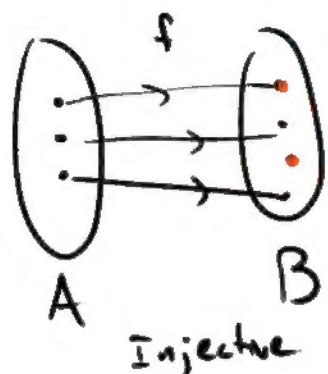
"range space" / "image".

WHY CARE ABOUT THESE SPACES?

INJECTIVITY AND SURJECTIVITY

Defⁿ: Let $f: A \rightarrow B$ be a function. We say f is injective (or one-to-one) when for all $x, y \in A$, $f(x) = f(y)$ implies $x = y$.

Pictures:



NB: The kernel of a transformation should tell us something about injectivity...

i.e. $\text{Ker}(L) = \{v \in V : L(v) = 0_w\}$

So if $\text{Ker}(L) \neq \{0_v\}$, then $x \in \text{Ker}(L)$ w/ $x \neq 0_v$ but $L(x) = 0_w = L(0_v)$.

If $\text{Ker}(L) \neq \{0_v\}$, then L is not injective.

On the other hand, If L is not injective, then there are $u, v \in V$ w/ $\underline{L(u) = L(v)}$ but $u \neq v$.

Now $L(u-v) = L(u) - L(v) = 0_w$, but

$u \neq v$ implies $u-v \neq 0_v$. Thus, $\text{Ker}(L) \neq \{0_v\}$.

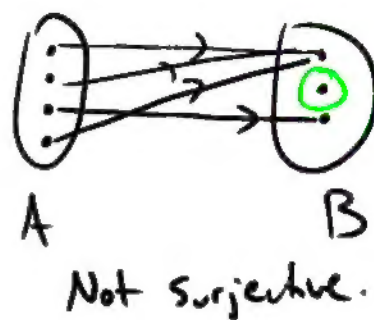
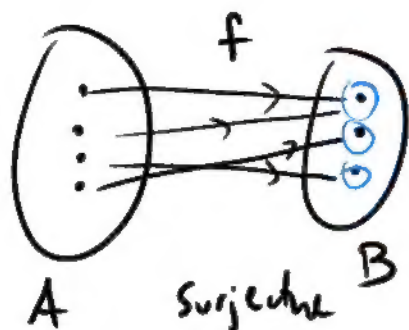
Prop: Let $L: V \rightarrow W$ be a linear map. L is injective if and only if $\text{Ker}(L) = \{0_v\}$. pf: Above \smile \square

Ex: $L(c + bx + ax^2) = \begin{pmatrix} 3a-b & 2b+c \\ a-c & a+b+c \end{pmatrix}$ is injective from earlier work $\smile \square$

Q: Which of the maps we discussed today were injective?

Defⁿ: A function $f: A \rightarrow B$ is surjective (or onto) when for all $b \in B$ there is $a \in A$ w/ $f(a) = b$.

Picture:



Ex: $L\left(\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}\right) = \begin{pmatrix} x+y+z \\ x-y+w \end{pmatrix}$ is surjective.

because $\text{ran}(L) \supseteq \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} = E_2$,

\uparrow \uparrow
 $x=y=z=0, w=1$ $x=y=z=0, w=1$

we see $\mathbb{R}^2 = \text{span}(E_2) \subseteq \text{ran}(L) \subseteq \mathbb{R}^2$. □

NB: If $\text{ran}(L) = \text{cod}(L) = W$ (where $L: V \rightarrow W$), then L is surjective (by definition). If L is surjective, then $\text{ran}(L) = \{L(v) : v \in V\} = W$ b/c every vector $w \in W$ is $L(v) = w$ for some $v \in V$.

Prop: The linear map $L: V \rightarrow W$ is surjective if and only if $\text{ran}(L) = W$.

Q: What if L is both surjective and injective?
 \hookrightarrow " L is bijective" $\leadsto L$ is a "linear isomorphism".